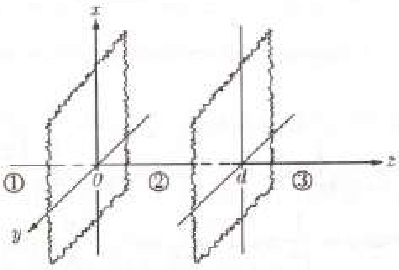
Problem 9.34



$$z<0: \quad \begin{cases} \tilde{\mathbf{E}}_I(z,t)=\tilde{E}_I e^{i(k_1z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_I(z,t)=\frac{1}{v_1}\tilde{E}_I e^{i(k_1z-\omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_R(z,t)=\tilde{E}_R e^{i(-k_1z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_R(z,t)=-\frac{1}{v_1}\tilde{E}_R e^{i(-k_1z-\omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$0<z<d: \quad \begin{cases} \tilde{\mathbf{E}}_r(z,t)=\tilde{E}_r e^{i(k_2z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_r(z,t)=\frac{1}{v_2}\tilde{E}_r e^{i(k_2z-\omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_l(z,t)=\tilde{E}_l e^{i(-k_2z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_l(z,t)=-\frac{1}{v_2}\tilde{E}_l e^{i(-k_2z-\omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$z>d: \quad \left\{ \tilde{\mathbf{E}}_T(z,t)=\tilde{E}_T e^{i(k_3z-\omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t)=\frac{1}{v_3}\tilde{E}_T e^{i(k_3z-\omega t)}\,\hat{\mathbf{y}}. \right.$$

Boundary conditions: $\mathbf{E}_1^{\parallel}=\mathbf{E}_2^{\parallel}$, $\mathbf{B}_1^{\parallel}=\mathbf{B}_2^{\parallel}$, at each boundary (assuming $\mu_1=\mu_2=\mu_3=\mu_0$):

$$z=0: \quad \begin{cases} \tilde{E}_I+\tilde{E}_R=\tilde{E}_r+\tilde{E}_l; \\ \\ \dfrac{1}{v_1}\tilde{E}_I-\dfrac{1}{v_1}\tilde{E}_R=\dfrac{1}{v_2}\tilde{E}_r-\dfrac{1}{v_2}\tilde{E}_l \Rightarrow \tilde{E}_I-\tilde{E}_R=\beta(\tilde{E}_r-\tilde{E}_l), \text{ where } \beta\equiv v_1/v_2. \end{cases}$$

$$z=d: \quad \begin{cases} \tilde{E}_r e^{ik_2d}+\tilde{E}_l e^{-ik_2d}=\tilde{E}_T e^{ik_3d}; \\ \\ \dfrac{1}{v_2}\tilde{E}_r e^{ik_2d}-\dfrac{1}{v_2}\tilde{E}_l e^{-ik_2d}=\dfrac{1}{v_3}\tilde{E}_T e^{ik_3d} \Rightarrow \tilde{E}_r e^{ik_2d}-\tilde{E}_l e^{-ik_2d}=\alpha\tilde{E}_T e^{ik_3d}, \text{ where } \alpha\equiv v_2/v_3. \end{cases}$$

We have here four equations; the problem is to eliminate $\tilde{E}_R$, $\tilde{E}_r$, and $\tilde{E}_l$, to obtain a single equation for $\tilde{E}_T$ in terms of $\tilde{E}_I$.

Add the first two to eliminate $\tilde{E}_R$: $\quad 2\tilde{E}_I=(1+\beta)\tilde{E}_r+(1-\beta)\tilde{E}_l$;
Add the last two to eliminate $\tilde{E}_l$: $\quad 2\tilde{E}_r e^{ik_2d}=(1+\alpha)\tilde{E}_T e^{ik_3d}$;
Subtract the last two to eliminate $\tilde{E}_r$: $\quad 2\tilde{E}_l e^{-ik_2d}=(1-\alpha)\tilde{E}^T e^{ik_3d}$.

Plug the last two of these into the first:

$$\begin{aligned} 2\tilde{E}_I &= (1+\beta)\frac{1}{2}e^{-ik_2d}(1+\alpha)\tilde{E}_T e^{ik_3d}+(1-\beta)\frac{1}{2}e^{ik_2d}(1-\alpha)\tilde{E}_T e^{ik_3d} \\ 4\tilde{E}_I &= \left[(1+\alpha)(1+\beta)e^{-ik_2d}+(1-\alpha)(1-\beta)e^{ik_2d}\right]\tilde{E}_T e^{ik_3d} \\ &= \left[(1+\alpha\beta)\left(e^{-ik_2d}+e^{ik_2d}\right)+(\alpha+\beta)\left(e^{-ik_2d}-e^{ik_2d}\right)\right]\tilde{E}_T e^{ik_3d} \\ &= 2\left[(1+\alpha\beta)\cos(k_2d)-i(\alpha+\beta)\sin(k_2d)\right]\tilde{E}_T e^{ik_3d}. \end{aligned}$$

Now the transmission coefficient is $T = \frac{v_3\epsilon_3 E_{T_0}^2}{v_1\epsilon_1 E_{I_0}^2} = \frac{v_3}{v_1}\left(\frac{\mu_0\epsilon_3}{\mu_0\epsilon_1}\right)\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \frac{v_1}{v_3}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \alpha\beta\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2}$, so

$$\begin{aligned}
T^{-1} &= \frac{1}{\alpha\beta}\frac{|\tilde{E}_I|^2}{|\tilde{E}_T|^2} = \frac{1}{\alpha\beta}\left|\frac{1}{2}\left[(1+\alpha\beta)\cos(k_2 d) - i(\alpha+\beta)\sin(k_2 d)\right]e^{ik_3 d}\right|^2 \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2\cos^2(k_2 d) + (\alpha+\beta)^2\sin^2(k_2 d)\right]. \quad \text{But } \cos^2(k_2 d) = 1-\sin^2(k_2 d). \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 + (\alpha^2+2\alpha\beta+\beta^2-1-2\alpha\beta-\alpha^2\beta^2)\sin^2(k_2 d)\right] \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)\right].
\end{aligned}$$

But $n_1 = \frac{c}{v_1}$, $n_2 = \frac{c}{v_2}$, $n_3 = \frac{c}{v_3}$, so $\alpha = \frac{n_3}{n_3}$, $\beta = \frac{n_2}{n_1}$.

$$= \boxed{\frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2(k_2 d)\right].}$$

Problem 9.35

$T = 1 \Rightarrow \sin kd = 0 \Rightarrow kd = 0, \pi, 2\pi \ldots.$ The *minimum* (nonzero) thickness is $d = \pi/k$. But $k = \omega/v = 2\pi\nu/v = 2\pi\nu n/c$, and $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ (Eq. 9.69), where (presumably) $\mu \approx \mu_0$. So $n = \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$, and hence $d = \frac{\pi c}{2\pi\nu\sqrt{\epsilon_r}} = \frac{c}{2\nu\sqrt{\epsilon_r}} = \frac{3\times 10^8}{2(10\times 10^9)\sqrt{2.5}} = 9.49\times 10^{-3}$ m, or $\boxed{9.5\,\text{mm.}}$

Problem 9.36

From Eq. 9.199,

$$\begin{aligned}
T^{-1} &= \frac{1}{4(4/3)(1)}\left\{[(4/3)+1]^2 + \frac{[(16/9)-(9/4)][1-(9/4)]}{(9/4)}\sin^2(3\omega d/2c)\right\} \\
&= \frac{3}{16}\left[\frac{49}{9} + \frac{(-17/36)(-5/4)}{(9/4)}\sin^2(3\omega d/2c)\right] = \frac{49}{48} + \frac{85}{(48)(36)}\sin^2(3\omega d/2c). \\
T &= \frac{48}{49+(85/36)\sin^2(3\omega d/2c)}.
\end{aligned}$$

Since $\sin^2(3\omega d/2c)$ ranges from 0 to 1, $T_{\min} = \frac{48}{49+(85/36)} = \boxed{0.935;}$ $T_{\max} = \frac{48}{49} = \boxed{0.980.}$ Not much variation, and the transmission is good (over 90%) for *all* frequencies. Since Eq. 9.199 is unchanged when you switch 1 and 3, the transmission is the same either direction, and the $\boxed{\text{fish sees you just as well as you see it.}}$

Problem 10.1

$$\begin{aligned}
\Box^2 V + \frac{\partial L}{\partial t} &= \nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) + \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = \nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{1}{\epsilon_0}\rho. \checkmark \\
\Box^2\mathbf{A} - \nabla L &= \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \checkmark
\end{aligned}$$

Problem 10.3

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.} \quad \mathbf{B} = \nabla\times\mathbf{A} = \boxed{0.}$$

This is a funny set of potentials for a $\boxed{\text{stationary point charge}}$ q at the origin. ($V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, $\mathbf{A} = 0$ would, of course, be the customary choice.) Evidently $\boxed{\rho = q\delta^3(\mathbf{r});\ \mathbf{J} = 0.}$

Problem 10.5

$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r}}; \quad \mathbf{A}' = \mathbf{A} + \nabla\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} + \left(-\frac{1}{4\pi\epsilon_0}qt\right)\left(-\frac{1}{r^2}\,\hat{\mathbf{r}}\right) = \boxed{0.}$$

This gauge function transforms the "funny" potentials of Prob. 10.3 into the "ordinary" potentials of a stationary point charge.

Problem 10.7

Suppose $\nabla\cdot\mathbf{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. (Let $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$—some known function.) We want to pick λ such that $\mathbf{A}'$ and V' (Eq. 10.7) *do* obey $\nabla\cdot\mathbf{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$.

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda.$$

This will be zero provided we pick for λ the solution to $\Box^2\lambda = -\Phi$, which by hypothesis (and in fact) we know how to solve.

We *could* always find a gauge in which $V' = 0$, simply by picking $\lambda = \int_0^t V\,dt'$. We *cannot* in general pick $\mathbf{A} = 0$—this would make $\mathbf{B} = 0$. [Finding such a gauge function would amount to expressing $\mathbf{A}$ as $-\nabla\lambda$, and we know that vector functions *cannot* in general be written as gradients—only if they happen to have curl zero, which $\mathbf{A}$ (ordinarily) does *not*.]

Problem 10.9

(a) As in Ex. 10.2, for $t < r/c$, $\mathbf{A} = 0$; for $t > r/c$,

$$\begin{aligned}
\mathbf{A}(r,t) &= \left(\frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\right) 2\int_0^{\sqrt{(ct)^2-r^2}} \frac{k(t-\sqrt{r^2+z^2}/c)}{\sqrt{r^2+z^2}}\,dz = \frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\left\{t\int_0^{\sqrt{(ct)^2-r^2}}\frac{dz}{\sqrt{r^2+z^2}} - \frac{1}{c}\int_0^{\sqrt{(ct)^2-r^2}} dz\right\} \\
&= \left(\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\right)\left[t\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) - \frac{1}{c}\sqrt{(ct)^2-r^2}\right]. \quad \text{Accordingly,}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(r,t) &= -\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\left\{\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \right. \\
&\qquad \left. t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\left(\frac{1}{r}\right)\left(c + \frac{1}{2}\frac{2c^2t}{\sqrt{(ct)^2-r^2}}\right) - \frac{1}{2c}\frac{2c^2t}{\sqrt{(ct)^2-r^2}}\right\} \\
&= -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\left\{\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \frac{ct}{\sqrt{(ct)^2-r^2}} - \frac{ct}{\sqrt{(ct)^2-r^2}}\right\} \\
&= \boxed{-\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{\mathbf{z}}} \qquad \text{(or zero, for } t < r/c\text{).}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}(r,t) &= -\frac{\partial A_z}{\partial r}\,\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 k}{2\pi}\left\{t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\frac{\left[r\frac{1}{2}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} - ct - \sqrt{(ct)^2-r^2}\right]}{r^2} - \frac{1}{2c}\frac{(-2r)}{\sqrt{(ct)^2-r^2}}\right\}\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 k}{2\pi}\left\{\frac{-ct^2}{r\sqrt{(ct)^2-r^2}} + \frac{r}{c\sqrt{(ct)^2-r^2}}\right\}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 k}{2\pi}\frac{(-c^2t^2+r^2)}{rc\sqrt{(ct)^2-r^2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 k}{2\pi rc}\sqrt{(ct)^2-r^2}\,\hat{\boldsymbol{\phi}}.}
\end{aligned}$$

(b) $\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{q_0\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz$. But $\mathscr{r} = \sqrt{r^2+z^2}$, so the integrand is even in z:

$$\mathbf{A}(r,t) = \left(\frac{\mu_0 q_0}{4\pi}\,\hat{\mathbf{z}}\right) 2\int_0^{\infty}\frac{\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz.$$

Now $z = \sqrt{\mathscr{r}^2 - r^2} \Rightarrow dz = \dfrac{1}{2}\dfrac{2\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2 - r^2}} = \dfrac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2 - r^2}}$, and $z = 0 \Rightarrow \mathscr{r} = r$, $z = \infty \Rightarrow \mathscr{r} = \infty$. So:

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}} \int_r^\infty \frac{1}{\mathscr{r}}\delta\left(t - \frac{\mathscr{r}}{c}\right)\frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2 - r^2}}.$$

Now $\delta(t - \mathscr{r}/c) = c\delta(\mathscr{r} - ct)$ (Ex. 1.15); therefore $\mathbf{A} = \dfrac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}}\,c\displaystyle\int_r^\infty \frac{\delta(\mathscr{r} - ct)}{\sqrt{\mathscr{r}^2 - r^2}}\,d\mathscr{r}$, so

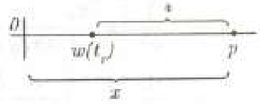
$$\mathbf{A}(r,t) = \frac{\mu_0 q_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2 - r^2}}\,\hat{\mathbf{z}} \quad \text{(or zero, if } ct < r\text{);}$$

$$\mathbf{E}(r,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{2c^2 t}{[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0 q_0 c^3 t}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\mathbf{z}}} \quad \text{(or zero, for } t < r/c\text{);}$$

$$\mathbf{B}(r,t) = -\frac{\partial \mathbf{A}_z}{\partial t}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{-2r}{[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{-\mu_0 q_0 c r}{2\pi[(ct)^2 - r^2]^{3/2}}\,\hat{\boldsymbol{\phi}}} \quad \text{(or zero, for } t < r/c\text{).}$$

Problem 10.16

First calculate t_r: $t_r = t - |\mathbf{r} - \mathbf{w}(t_r)|/c \Rightarrow$



$-c(t_r - t) = x - \sqrt{b^2 + c^2 t_r^2} \Rightarrow c(t_r - t) + x = \sqrt{b^2 + c^2 t_r^2}$;

$c^2 t_r^2 - 2c^2 t_r t + c^2 t^2 + 2xct_r - 2xct + x^2 = b^2 + c^2 t_r^2$;

$2ct_r(x - ct) + (x^2 - 2xct + c^2 t^2) = b^2$;

$2ct_r(x - ct) = b^2 - (x - ct)^2$, or $t_r = \dfrac{b^2 - (x - ct)^2}{2c(x - ct)}$.

Now $V(x,t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}$, and $\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \mathscr{r}(c - v)$; $\mathscr{r} = c(t - t_r)$.

$$v = \frac{1}{2}\frac{1}{\sqrt{b^2 + c^2 t_r^2}}2c^2 t_r = \frac{c^2 t_r}{c(t_r - t) + x} = \frac{c^2 t_r}{ct_r + (x - ct)};\ (c - v) = \frac{c^2 t_r + c(x - ct) - c^2 t_r}{ct_r + (x - ct)} = \frac{c(x - ct)}{ct_r + (x - ct)};$$

$$\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \frac{c(t - t_r)c(x - ct)}{ct_r + (x - ct)} = \frac{c^2(t - t_r)(x - ct)}{ct_r + (x - ct)};\ ct_r + (x - ct) = \frac{b^2 - (x - ct)^2}{2(x - ct)} + (x - ct) = \frac{b^2 + (x - ct)^2}{2(x - ct)};$$

$$t - t_r = \frac{2ct(x - ct) - b^2 + (x - ct)^2}{2c(x - ct)} = \frac{(x - ct)(x + ct) - b^2}{2c(x - ct)} = \frac{(x^2 - c^2t^2 - b^2)}{2c(x - ct)}.$$ Therefore

$$\frac{1}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} = \left[\frac{b^2 + (x - ct)^2}{2(x - ct)}\right]\frac{1}{c^2(x - ct)}\frac{2c(x - ct)}{[2ct(x - ct) - b^2 + (x - ct)^2]} = \frac{b^2 + (x - ct)^2}{c(x - ct)\,[2ct(x - ct) - b^2 + (x - ct)^2]}.$$

The term in square brackets simplifies to $(2ct + x - ct)(x - ct) - b^2 = (x + ct)(x - ct) - b^2 = x^2 - c^2t^2 - b^2$.

So $\boxed{V(x,t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}.}$

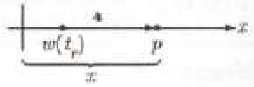
Meanwhile

$$\mathbf{A} = \frac{V}{c^2}\mathbf{v} = \frac{c^2 t_r}{ct_r + (x - ct)}\frac{V}{c^2}\,\hat{\mathbf{x}} = \left[\frac{b^2 - (x - ct)^2}{2c(x - ct)}\right]\frac{2(x - ct)}{b^2 + (x - ct)^2}\frac{q}{4\pi\epsilon_0}\frac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}}$$

$$= \boxed{\frac{q}{4\pi\epsilon_0 c}\frac{b^2 - (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}}.}$$

Problem 10.18

$\mathbf{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$. Here $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and, for points to the *right*, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{x}}$. So $\mathbf{u} = (c-v)\,\hat{\mathbf{x}}$, $\mathbf{u}\times\mathbf{a}=0$, and $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c-v)$.



$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2-v^2)(c-v)\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)^2}{(c-v)^3}\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}};\\
\mathbf{B} &= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} = 0. \quad \text{qed}
\end{aligned}$$

For field points to the *left*, $\hat{\boldsymbol{\mathscr{r}}} = -\hat{\mathbf{x}}$ and $\mathbf{u} = -(c+v)\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c+v)$, and

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2-v^2)(c+v)\,\hat{\mathbf{x}} = \boxed{\frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}};\ \mathbf{B}=0.}$$

Problem 10.23

Using Product Rule #5, Eq. 10.43 $\Rightarrow$

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-1/2}\\
&= \frac{\mu_0 qc}{4\pi}\mathbf{v}\cdot\left\{-\frac{1}{2}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\right\}\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left\{-2(c^2t-\mathbf{r}\cdot\mathbf{v})\nabla(\mathbf{r}\cdot\mathbf{v})+(c^2-v^2)\nabla(r^2)\right\}.
\end{aligned}$$

Product Rule #4 $\Rightarrow$

$$\begin{aligned}
\nabla(\mathbf{r}\cdot\mathbf{v}) &= \mathbf{v}\times(\nabla\times\mathbf{r})+(\mathbf{v}\cdot\nabla)\mathbf{r}, \text{ but } \nabla\times\mathbf{r}=0,\\
(\mathbf{v}\cdot\nabla)\mathbf{r} &= \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = v_x\,\hat{\mathbf{x}}+v_y\,\hat{\mathbf{y}}+v_z\,\hat{\mathbf{z}} = \mathbf{v}, \text{ and}\\
\nabla(r^2) &= \nabla(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\times(\nabla\times\mathbf{r})+2(\mathbf{r}\cdot\nabla)\mathbf{r} = 2\mathbf{r}. \text{ So}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left[-2(c^2t-\mathbf{r}\cdot\mathbf{v})\mathbf{v}+(c^2-v^2)2\mathbf{r}\right]\\
&= \frac{\mu_0 qc}{4\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left\{(c^2t-\mathbf{r}\cdot\mathbf{v})v^2-(c^2-v^2)(\mathbf{r}\cdot\mathbf{v})\right\}.
\end{aligned}$$

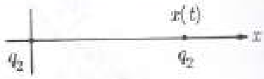
But the term in curly brackets is : $c^2tv^2 - v^2(\mathbf{r}\cdot\mathbf{v}) - c^2(\mathbf{r}\cdot\mathbf{v}) + v^2(\mathbf{r}\cdot\mathbf{v}) = c^2(v^2t-\mathbf{r}\cdot\mathbf{v})$.

$$= \frac{\mu_0 qc^3}{4\pi}\frac{(v^2t-\mathbf{r}\cdot\mathbf{v})}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}.$$

Meanwhile, from Eq. 10.42,

$$\begin{aligned}
-\mu_0\epsilon_0\frac{\partial V}{\partial t} &= -\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}qc\left(-\frac{1}{2}\right)\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\ \times\\
&\qquad \frac{\partial}{\partial t}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left[2(c^2t-\mathbf{r}\cdot\mathbf{v})c^2+(c^2-v^2)(-2c^2t)\right]\\
&= -\frac{\mu_0 qc^3}{4\pi}\frac{(c^2t-\mathbf{r}\cdot\mathbf{v}-c^2t+v^2t)}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}} = \nabla\cdot\mathbf{A}.\ \checkmark
\end{aligned}$$

Problem 10.24



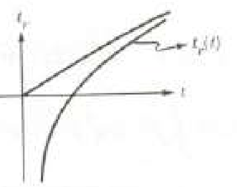
(a) $\boxed{\mathbf{F}_2 = \dfrac{q_1q_2}{4\pi\epsilon_0}\dfrac{1}{(b^2+c^2t^2)}\,\hat{\mathbf{x}}.}$

(This is just Coulomb's law, since q_1 is at rest.)

(b) $I_2 = \dfrac{q_1q_2}{4\pi\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\frac{1}{(b^2+c^2t^2)}\,dt = \frac{q_1q_2}{4\pi\epsilon_0}\left[\frac{1}{bc}\tan^{-1}(ct/b)\right]\Big|_{-\infty}^{\infty} = \frac{q_1q_2}{4\pi\epsilon_0 bc}\left[\tan^{-1}(\infty)-\tan^{-1}(-\infty)\right]$

$= \dfrac{q_1q_2}{4\pi\epsilon_0 bc}\left[\dfrac{\pi}{2}-\left(-\dfrac{\pi}{2}\right)\right] = \boxed{\dfrac{q_1q_2}{4\pi\epsilon_0}\dfrac{\pi}{bc}.}$

(c) From Prob. 10.18, $\mathbf{E} = -\frac{q_2}{4\pi\epsilon_0}\frac{1}{x^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}}$. Here x and v are to be evaluated at the retarded time t_r, which is given by $c(t-t_r) = x(t_r) = \sqrt{b^2+c^2t_r^2} \Rightarrow c^2t^2 - 2ctt_r + c^2t_r^2 = b^2 + c^2t_r^2 \Rightarrow t_r = \frac{c^2t^2-b^2}{2c^2t}$. *Note:* As we found in Prob. 10.15, q_2 first "comes into view" (for q_1) at time $t=0$. Before that it can exert no force on q_1, and there *is* no retarded time. From the graph of t_r versus t we see that t_r ranges all the way from $-\infty$ to ∞ while $t > 0$.



$$x(t_r) = c(t-t_r) = \frac{2c^2t^2 - c^2t^2 + b^2}{2ct} = \frac{b^2+c^2t^2}{2ct} \text{ (for } t>0). \quad v(t) = \frac{1}{2}\frac{2c^2t}{\sqrt{b^2+c^2t^2}} = \frac{c^2t}{x}, \text{ so}$$

$$v(t_r) = \left(\frac{c^2t^2-b^2}{2t}\right)\left(\frac{2ct}{b^2+c^2t^2}\right) = c\left(\frac{c^2t^2-b^2}{c^2t^2+b^2}\right) \text{ (for } t>0). \quad \text{Therefore}$$

$$\frac{c-v}{c+v} = \frac{(c^2t^2+b^2)-(c^2t^2-b^2)}{(c^2t^2+b^2)+(c^2t^2-b^2)} = \frac{2b^2}{2c^2t^2} = \frac{b^2}{c^2t^2} \text{ (for } t>0). \quad \mathbf{E} = -\frac{q_2}{4\pi\epsilon_0}\frac{4c^2t^2}{(b^2+c^2t^2)^2}\frac{b^2}{c^2t^2}\hat{\mathbf{x}} \Rightarrow$$

$$\boxed{\mathbf{F}_1 = \begin{cases} 0, & t<0; \\ -\frac{q_1q_2}{4\pi\epsilon_0}\frac{4b^2}{(b^2+c^2t^2)^2}\hat{\mathbf{x}}, & t>0. \end{cases}}$$

(d) $I_1 = -\frac{q_1q_2}{4\pi\epsilon_0}4b^2\int_0^\infty \frac{1}{(b^2+c^2t^2)^2}\,dt$. The integral is

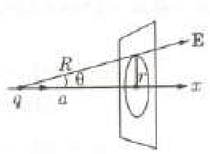
$$\frac{1}{c^4}\int_0^\infty \frac{1}{[(b/c)^2+t^2]^2}\,dt = \frac{1}{c^4}\left(\frac{c^2}{2b^2}\right)\left[\left.\frac{t}{(b/c)^2+t^2}\right|_0^\infty + \int_0^\infty \frac{1}{[(b/c)^2+t^2)]}\,dt\right] = \frac{1}{2c^2b^2}\left(\frac{\pi c}{2b}\right) = \frac{\pi}{4cb^3}.$$

So $\boxed{I_1 = -\frac{q_1q_2}{4\pi\epsilon_0}\frac{\pi}{bc}.}$

(e) $\mathbf{F}_1 \neq -\mathbf{F}_2$, so Newton's third law is *not* obeyed. On the other hand, $I_1 = -I_2$ in this instance, which suggests that the *net* momentum delivered from (1) to (2) is equal and opposite to the net momentum delivered from (2) to (1), and hence that the total mechanical momentum is conserved. (In general, the fields might carry off some momentum, leaving the mechanical momentum altered; but that doesn't happen in the present case.)

Problem 10.25

$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$; $\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$ (Eq. 10.69).



So $\mathbf{S} = \frac{1}{\mu_0 c^2}\left[\mathbf{E} \times (\mathbf{v} \times \mathbf{E})\right] = \epsilon_0 \left[E^2\mathbf{v} - (\mathbf{v} \cdot \mathbf{E})\mathbf{E}\right]$.

The power crossing the plane is $P = \int \mathbf{S} \cdot d\mathbf{a}$,

and $d\mathbf{a} = 2\pi r\, dr\, \hat{\mathbf{x}}$ (see diagram). So

$$\begin{aligned}
P &= \epsilon_0 \int (E^2 v - E_x^2 v) 2\pi r\, dr; \quad E_x = E\cos\theta, \text{ so } E^2 - E_x^2 = E^2 \sin^2\theta. \\
&= 2\pi\epsilon_0 v \int E^2 \sin^2\theta\, r\, dr. \quad \text{From Eq. 10.68, } \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2}\frac{\hat{\mathbf{R}}}{R^2\left[1-(v/c)^2\sin^2\theta\right]^{3/2}} \text{ where } \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}. \\
&= 2\pi\epsilon_0 v \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{\gamma^2}\int_0^\infty \frac{r\sin^2\theta}{R^4\left[1-(v/c)^2\sin^2\theta\right]^3}\, dr. \quad \text{Now } r = a\tan\theta \Rightarrow dr = a\frac{1}{\cos^2\theta}\,d\theta;\ \frac{1}{R} = \frac{\cos\theta}{a}. \\
&= \frac{v}{2\gamma^4}\frac{q^2}{4\pi\epsilon_0}\frac{1}{a^2}\int_0^{\pi/2} \frac{\sin^3\theta\cos\theta}{\left[1-(v/c)^2\sin^2\theta\right]^3}\, d\theta. \quad \text{Let } u \equiv \sin^2\theta, \text{ so } du = 2\sin\theta\cos\theta\, d\theta. \\
&= \frac{vq^2}{16\pi\epsilon_0 a^2\gamma^4}\int_0^1 \frac{u}{\left[1-(v/c)^2 u\right]^3}\, du = \frac{vq^2}{16\pi\epsilon_0 a^2\gamma^4}\left(\frac{\gamma^4}{2}\right) = \boxed{\frac{vq^2}{32\pi\epsilon_0 a^2}}.
\end{aligned}$$

Problem 11.3

$P = I^2 R = q_0^2\omega^2\sin^2(\omega t)R$ (Eq. 11.15) $\Rightarrow \langle P\rangle = \frac{1}{2}q_0^2\omega^2 R$. Equate this to Eq. 11.22:

$\frac{1}{2}q_0^2\omega^2 R = \frac{\mu_0 q_0^2 d^2\omega^4}{12\pi c} \Rightarrow \boxed{R = \frac{\mu_0 d^2\omega^2}{6\pi c};}$ or, since $\omega = \frac{2\pi c}{\lambda}$,

$$R = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2c^2}{\lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi(4\pi\times10^{-7})(3\times10^8)\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2\,\Omega = \boxed{789.6(d/\lambda)^2\,\Omega.}$$

For the wires in an ordinary radio, with $d = 5\times10^{-2}$ m and (say) $\lambda = 10^3$ m, $R = 790(5\times10^{-5})^2 = 2\times10^{-6}\,\Omega$, which is negligible compared to the Ohmic resistance.

Problem 11.5

Go back to Eq. 11.33:

$$\mathbf{A} = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.$$

Since $V = 0$ here,

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}(-\omega)\sin[\omega(t-r/c)] - \frac{\omega}{c}\omega\cos[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}} \\
&= \boxed{\frac{\mu_0 m_0 \omega}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\sin[\omega(t-r/c)] + \frac{\omega}{c}\cos[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.} \\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 m_0}{4\pi}\left\{\frac{1}{r\sin\theta}\frac{2\sin\theta\cos\theta}{r}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{\mathbf{r}}\right. \\
&\quad \left. - \frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] - \frac{\omega}{c}\left(-\frac{\omega}{c}\right)\cos[\omega(t-r/c)]\right]\hat{\boldsymbol{\theta}}\right\} \\
&= \boxed{\begin{aligned}\frac{\mu_0 m_0}{4\pi}\ &\left\{\frac{2\cos\theta}{r^2}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{\mathbf{r}}\right. \\
&\left. -\frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] + \left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)]\right]\hat{\boldsymbol{\theta}}\right\}.\end{aligned}}
\end{aligned}$$

These are precisely the fields we studied in Prob. 9.33, with $A \to \dfrac{\mu_0 m_0 \omega^2}{4\pi c}$. The Poynting vector (quoting the solution to that problem) is

$$\boxed{\begin{aligned}\mathbf{S} \;=\; &\frac{\mu_0 m_0^2\omega^3}{16\pi^2 c^2}\left(\frac{\sin\theta}{r^2}\right)\left\{\frac{2\cos\theta}{r}\left[\left(1-\frac{c^2}{\omega^2 r^2}\right)\sin u\cos u + \frac{c}{\omega r}\left(\cos^2 u - \sin^2 u\right)\right]\hat{\boldsymbol{\theta}}\right. \\
&\left.\sin\theta\left[\left(-\frac{2}{r}+\frac{c^2}{\omega^2 r^3}\right)\sin u\cos u + \frac{\omega}{c}\cos^2 u + \frac{c}{\omega r^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{\mathbf{r}}\right\},\end{aligned}}$$

where $u \equiv -\omega(t-r/c)$. The intensity is $\boxed{\langle\mathbf{S}\rangle = \dfrac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\dfrac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}},}$ the same as Eq. 11.39.

Problem 11.14

$F = \dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{r^2} = ma = m\dfrac{v^2}{r} \Rightarrow v = \sqrt{\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{mr}}$. At the beginning ($r_0 = 0.5$ Å),

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\times10^{-11})}\right]^{-1/2}\frac{1}{3\times10^8} = 0.0075,$$

and when the radius is one hundredth of this v/c is only 10 times greater (0.075), so for *most* of the trip the velocity is safely nonrelativistic.

¿From the Larmor formula, $P = \dfrac{\mu_0 q^2}{6\pi c}\left(\dfrac{v^2}{r}\right)^2 = \dfrac{\mu_0 q^2}{6\pi c}\left(\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{mr^2}\right)^2$ (since $a = v^2/r$), and $P = -dU/dt$, where U is the (total) energy of the electron:

$$U = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{r}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r}.$$

So $-\dfrac{dU}{dt} = -\dfrac{1}{8\pi\epsilon_0}\dfrac{q^2}{r^2}\dfrac{dr}{dt} = P = \dfrac{q^2}{6\pi\epsilon_0 c^3}\left(\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{mr^2}\right)^2$, and hence $\dfrac{dr}{dt} = -\dfrac{1}{3c}\left(\dfrac{q^2}{2\pi\epsilon_0 mc}\right)^2\dfrac{1}{r^2}$, or

$dt = -3c\left(\dfrac{2\pi\epsilon_0 mc}{q^2}\right)^2 r^2\,dr \Rightarrow t = -3c\left(\dfrac{2\pi\epsilon_0 mc}{q^2}\right)^2\displaystyle\int_{r_0}^{0} r^2\,dr = \boxed{c\left(\dfrac{2\pi\epsilon_0 mc}{q^2}\right)^2 r_0^3}$

$= (3\times10^8)\left[\dfrac{2\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)}{(1.6\times10^{-19})^2}\right]^2(5\times10^{-11})^3 = \boxed{1.3\times10^{-11}\text{ s.}}$ (Not very long!)

Problem 11.15

Problem 11.17

(a) To counteract the radiation reaction (Eq. 11.80), you must exert a force $\mathbf{F}_e = -\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$. *For circular motion,* $\mathbf{r}(t) = R\left[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}\right]$, $\mathbf{v}(t) = \dot{\mathbf{r}} = R\omega\left[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}\right]$;

$\mathbf{a}(t) = \dot{\mathbf{v}} = -R\omega^2\left[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}\right] = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v}.}$

$\boxed{P_e = \mathbf{F}_e\cdot\mathbf{v} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ This is the power *you* must supply.

Meanwhile, the power *radiated* is (Eq. 11.70) $P_{\rm rad} = \frac{\mu_0 q^2 a^2}{6\pi c}$, and $a^2 = \omega^4 r^2 = \omega^4 R^2 = \omega^2 v^2$, so $P_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2$, and the two expressions agree.

(b) *For simple harmonic motion,* $\mathbf{r}(t) = A\cos(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{v} = \dot{\mathbf{r}} = -A\omega\sin(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{a} = \dot{\mathbf{v}} = -A\omega^2\cos(\omega t)\,\hat{\mathbf{z}} = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v};\ P_e = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ But this time $a^2 = \omega^4 r^2 = \omega^4 A^2\cos^2(\omega t)$, whereas $\omega^2 v^2 = \omega^4 A^2 \sin^2(\omega t)$, so

$$P_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2\cos^2(\omega t) \neq P_e = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2 \sin^2(\omega t);$$

the power you deliver is *not* equal to the power radiated. However, since the time *averages* of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ are equal (to wit: 1/2), *over a full cycle* the energy radiated is the same as the energy input. (In the mean time energy is evidently being stored temporarily in the nearby fields.)

(c) *In free fall,* $\mathbf{v}(t) = \frac{1}{2}gt^2\,\hat{\mathbf{y}}$; $\mathbf{v} = gt\,\hat{\mathbf{y}}$; $\mathbf{a} = g\,\hat{\mathbf{y}}$; $\dot{\mathbf{a}} = 0$. So $\boxed{\mathbf{F}_e = 0;}$ the radiation reaction is zero, and hence $\boxed{P_e = 0.}$ But there *is* radiation: $\boxed{P_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}g^2.}$ Evidently energy is being continuously extracted from the nearby fields. This paradox persists even in the *exact* solution (where we do *not* assume $v \ll c$, as in the Larmor formula and the Abraham-Lorentz formula)—see Prob. 11.31.

Problem 11.21

(a) This is an oscillating electric dipole, with amplitude $p_0 = qd$ and frequency $\omega = \sqrt{k/m}$. The (averaged) Poynting vector is given by Eq. 11.21: $\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$, so the power per unit area of *floor* is

$$I_f = \langle\mathbf{S}\rangle\cdot\hat{\mathbf{z}} = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta\cos\theta}{r^2}.\quad \text{But } \sin\theta = \frac{R}{r},\ \cos\theta = \frac{h}{r},\ \text{and } r^2 = R^2 + h^2.$$

$$= \boxed{\left(\frac{\mu_0 q^2 d^2\omega^4}{32\pi^2 c}\right)\frac{R^2 h}{(R^2+h^2)^{5/2}}.}$$

$\frac{dI_f}{dR} = 0 \Rightarrow \frac{d}{dR}\left[\frac{R^2}{(R^2+h^2)^{5/2}}\right] = 0 \Rightarrow \frac{2R}{(R^2+h^2)^{5/2}} - \frac{5}{2}\frac{R^2}{(R^2+h^2)^{7/2}}2R = 0 \Rightarrow$

$(R^2+h^2) - \frac{5}{2}R^2 = 0 \Rightarrow h^2 = \frac{3}{2}R^2 \Rightarrow \boxed{R = \sqrt{2/3}h,}$ for maximum intensity.

(b)

$$P = \int I_f(R)\,da = \int I_f(R)\,2\pi R\,dR = 2\pi\left(\frac{\mu_0 (qd)^2\omega^4}{32\pi^2 c}\right)h\int_0^\infty \frac{R^3}{(R^2+h^2)^{5/2}}\,dR.\quad \text{Let } x \equiv R^2:$$

$$\int_0^\infty \frac{R^3}{(R^2+h^2)^{5/2}}\,dR = \frac{1}{2}\int_0^\infty \frac{x}{(x+h^2)^{5/2}}\,dx = \frac{1}{2h}\frac{\Gamma(2)\Gamma(1/2)}{\Gamma(5/2)} = \frac{2}{3h}.$$

$$= 2\pi\left(\frac{\mu_0 q^2 d^2\omega^4}{32\pi^2 c}\right)h\frac{2}{3h} = \boxed{\frac{\mu_0 q^2 d^2\omega^4}{24\pi c},}$$

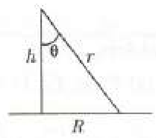
which should be (and *is*) half the *total* radiated power (Eq. 11.22)—the rest hits the ceiling, of course.

(c) The amplitude is $x_0(t)$, so $U = \frac{1}{2}kx_0^2$ is the energy, at time t, and $dU/dt = -2P$ is the power radiated:

$\frac{1}{2}k\frac{d}{dt}(x_0^2) = -\frac{\mu_0\omega^4}{12\pi c}q^2 x_0^2 \Rightarrow \frac{d}{dt}(x_0^2) = -\frac{\mu_0\omega^4 q^2}{6\pi kc}(x_0^2) = -\kappa x_0^2 \Rightarrow x_0^2 = d^2 e^{-\kappa t}$ or $x_0(t) = de^{-\kappa t/2}$.

$\tau = \frac{2}{\kappa} = \frac{12\pi kc}{\mu_0 q^2 k^2}m^2 = \boxed{\frac{12\pi cm^2}{\mu_0 q^2 k}.}$

Problem 11.22



(a) From Eq. 11.39, $\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}}$. Here $\sin\theta = R/r$, $r = \sqrt{R^2 + h^2}$, and the total radiated power (Eq. 11.40) is $P = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$. So the intensity is $I(R) = \left(\frac{12P}{32\pi} \right) \frac{R^2}{(R^2+h^2)^2} = \boxed{\frac{3P}{8\pi} \frac{R^2}{(R^2+h^2)^2}}$.

(b) The intensity *directly* below the antenna ($R = 0$) would (ideally) have been *zero*. The engineer *should* have measured it at the position of *maximum* intensity:

$$\frac{dI}{dR} = \frac{3P}{8\pi} \left[\frac{2R}{(R^2+h^2)^2} - \frac{2R^2}{(R^2+h^2)^3} 2R \right] = \frac{3P}{8\pi} \frac{2R}{(R^2+h^2)^3} (R^2 + h^2 - 2R^2) = 0 \Rightarrow \boxed{R = h.}$$

At this location the intensity is $I(h) = \frac{3P}{8\pi} \frac{h^2}{(2h^2)^2} = \boxed{\frac{3P}{32\pi h^2}}$.

(c) $I_{\max} = \frac{3(35 \times 10^3)}{32\pi(200)^2} = 0.026\,\mathrm{W/m^2} = \boxed{2.6\,\mu\mathrm{W/cm^2}}$. $\boxed{\text{Yes, KRUD is in compliance.}}$