Problem 7.1

(a) Let Q be the charge on the inner shell. Then $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$ in the space between them, and $(V_a - V_b) = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{4\pi\epsilon_0}Q\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$.

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{Q}{\epsilon_0} = \frac{\sigma}{\epsilon_0}\frac{4\pi\epsilon_0(V_a - V_b)}{(1/a - 1/b)} = \boxed{4\pi\sigma\frac{(V_a - V_b)}{(1/a - 1/b)}.}$$

(b) $R = \dfrac{V_a - V_b}{I} = \boxed{\dfrac{1}{4\pi\sigma}\left(\dfrac{1}{a}-\dfrac{1}{b}\right).}$

(c) For large b ($b \gg a$), the second term is negligible, and $R = 1/4\pi\sigma a$. Essentially all of the resistance is in the region right around the inner sphere. Successive shells, as you go out, contribute less and less, because the cross-sectional area ($4\pi r^2$) gets larger and larger. For the two submerged spheres, $R = \frac{2}{4\pi\sigma a} = \frac{1}{2\pi\sigma a}$ (one R as the current leaves the first, one R as it converges on the second). Therefore $I = V/R = \boxed{2\pi\sigma a V.}$

Problem 7.2

(a) $V = Q/C = IR$. Because positive I means the charge on the capacitor is *decreasing*, $\dfrac{dQ}{dt} = -I = -\dfrac{1}{RC}Q$, so $Q(t) = Q_0 e^{-t/RC}$. But $Q_0 = Q(0) = CV_0$, so $\boxed{Q(t) = CV_0 e^{-t/RC}.}$

Hence $I(t) = -\dfrac{dQ}{dt} = CV_0\dfrac{1}{RC}e^{-t/RC} = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(b) $W = \boxed{\frac{1}{2}CV_0^2.}$ The energy delivered to the resistor is $\displaystyle\int_0^\infty P\,dt = \int_0^\infty I^2R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty = \frac{1}{2}CV_0^2.$ ✓

(c) $V_0 = Q/C + IR$. This time positive I means Q is *increasing*: $\dfrac{dQ}{dt} = I = \dfrac{1}{RC}(CV_0 - Q) \Rightarrow \dfrac{dQ}{Q - CV_0} = -\dfrac{1}{RC}dt \Rightarrow \ln(Q - CV_0) = -\dfrac{1}{RC}t + \text{constant} \Rightarrow Q(t) = CV_0 + ke^{-t/RC}$. But $Q(0) = 0 \Rightarrow k = -CV_0$, so $\boxed{Q(t) = CV_0\left(1 - e^{-t/RC}\right).}$ $I(t) = \dfrac{dQ}{dt} = CV_0\left(\dfrac{1}{RC}e^{-t/RC}\right) = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(d) Energy from battery: $\displaystyle\int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}dt = \frac{V_0^2}{R}\left(-RCe^{-t/RC}\right)\Big|_0^\infty = \frac{V_0^2}{R}RC = \boxed{CV_0^2.}$

Since $I(t)$ is the same as in (a), the energy delivered to the resistor is again $\boxed{\frac{1}{2}CV_0^2.}$ The final energy in the capacitor is also $\boxed{\frac{1}{2}CV_0^2,}$ so $\boxed{\text{half}}$ the energy from the battery goes to the capacitor, and the other half to the resistor.

Problem 7.8

(a) The field of long wire is $\mathbf{B} = \dfrac{\mu_0 I}{2\pi s}\hat{\phi}$, so $\displaystyle\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0 I}{2\pi}\int_s^{s+a}\frac{1}{s}(a\,ds) = \boxed{\frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right).}$

(b) $\mathcal{E} = -\dfrac{d\Phi}{dt} = -\dfrac{\mu_0 I a}{2\pi}\dfrac{d}{dt}\ln\left(\dfrac{s+a}{s}\right)$, and $\dfrac{ds}{dt} = v$, so $-\dfrac{\mu_0 I a}{2\pi}\left(\dfrac{1}{s+a}\dfrac{ds}{dt} - \dfrac{1}{s}\dfrac{ds}{dt}\right) = \boxed{\dfrac{\mu_0 I a^2 v}{2\pi s(s+a)}.}$

The field points *out* of the page, so the force on a charge in the nearby side of the square is *to the right*. In the far side it's also to the right, but here the field is weaker, so the current flows $\boxed{\text{counterclockwise.}}$

(c) This time the flux is *constant*, so $\boxed{\mathcal{E} = 0.}$

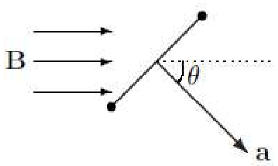
Problem 7.10

$\Phi = \mathbf{B}\cdot\mathbf{a} = Ba^2\cos\theta$

Here $\theta = \omega t$, so

$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2(-\sin\omega t)\omega;$

$\boxed{\mathcal{E} = B\omega a^2\sin\omega t.}$



(view from above)

Problem 7.18

$$\Phi = \int \mathbf{B}\cdot d\mathbf{a};\ \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi};\ \Phi = \frac{\mu_0 I a}{2\pi}\int_s^{s+a}\frac{ds'}{s'} = \frac{\mu_0 I a}{2\pi}\ln\frac{s+a}{s};$$

$$\mathcal{E} = I_{\text{loop}}R = \frac{dQ}{dt}R = -\frac{d\Phi}{dt} = -\frac{\mu_0 a}{2\pi}\ln(1+a/s)\frac{dI}{dt}.$$

$$dQ = -\frac{\mu_0 a}{2\pi R}\ln(1+a/s)\,dI \ \Rightarrow\ \boxed{Q = \frac{I\mu_0 a}{2\pi R}\ln(1+a/s).}$$

The field of the wire, at the square loop, is *out of the page*, and *decreasing*, so the field of the induced current must point out of page, within the loop, and hence the induced current flows $\boxed{\text{counterclockwise.}}$

Problem 7.20

(a) From Eq. 5.38, the field (on the axis) is $\mathbf{B} = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\hat{\mathbf{z}}$, so the flux through the little loop (area πa^2) is $\boxed{\Phi = \dfrac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

(b) The field (Eq. 5.86) is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$, where $m = I\pi a^2$. Integrating over the spherical "cap" (bounded by the big loop and centered at the little loop):

$$\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\int(2\cos\theta)(r^2\sin\theta\,d\theta\,d\phi) = \frac{\mu_0 I a^2}{2r}2\pi\int_0^{\bar{\theta}}\cos\theta\sin\theta\,d\theta$$

where $r = \sqrt{b^2+z^2}$ and $\sin\bar{\theta} = b/r$. Evidently $\Phi = \frac{\mu_0 I\pi a^2}{r}\left.\frac{\sin^2\theta}{2}\right|_0^{\bar{\theta}} = \boxed{\dfrac{\mu_0\pi I a^2 b^2}{2(b^2+z^2)^{3/2}},}$ the same as in (a)!!

(c) Dividing off I ($\Phi_1 = M_{12}I_2$, $\Phi_2 = M_{21}I_1$): $\boxed{M_{12} = M_{21} = \dfrac{\mu_0\pi a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

Problem 7.49

Initially, $\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2} \Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}$. After the magnetic field is on, the electron circles in a new orbit, of radius r_1 and velocity v_1:

$$\frac{mv_1^2}{r_1} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1^2} + qv_1B \Rightarrow T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1} + \frac{1}{2}qv_1r_1B.$$

But $r_1 = r + dr$, so $(r_1)^{-1} = r^{-1}\left(1+\frac{dr}{r}\right)^{-1} \cong r^{-1}\left(1-\frac{dr}{r}\right)$, while $v_1 = v + dv$, $B = dB$. To first order, then,

$$T_1 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}\left(1-\frac{dr}{r}\right) + \frac{1}{2}q(vr)\,dB, \text{ and hence } dT = T_1 - T = \frac{qvr}{2}\,dB - \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,dr.$$

Now, the induced electric field is $E = \frac{r}{2}\frac{dB}{dt}$ (Ex. 7.7), so $m\frac{dv}{dt} = qE = \frac{qr}{2}\frac{dB}{dt}$, or $m\,dv = \frac{qr}{2}dB$. The increase in kinetic energy is therefore $dT = d(\frac{1}{2}mv^2) = mv\,dv = \frac{qvr}{2}dB$. Comparing the two expressions, I conclude that $dr = 0$. qed

force in the z direction (a repulsion of the plates) which reduces their (electrical) attraction but does not deliver (horizontal) momentum to the plates.]

(ii) Meanwhile, in the space immediately above the upper plate the magnetic field drops abruptly to zero (as the plate moves past), inducing an *electric* field by Faraday's law. The magnetic field in the vicinity of the top plate (at $d(t) = d_0 - ut$) can be written, using Problem 1.46(b),

$$\mathbf{B}(z,t) = -\mu_0\sigma v\,\theta(d-z)\,\hat{\mathbf{x}}, \quad \Rightarrow \quad \frac{\partial\mathbf{B}}{\partial t} = \mu_0\sigma vu\,\delta(d-z)\,\hat{\mathbf{x}}.$$

In the analogy at the beginning of Section 7.2.2, the Faraday-induced electric field is just like the magnetostatic field of a surface current $\mathbf{K} = -\sigma vu\,\hat{\mathbf{x}}$. Referring to Eq. 5.58, then,

(I dropped the subscript on d_0, reverting to the original notation: d is the initial separation of the plates.)

The total impulse is thus $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \boxed{dA\mu_0\sigma^2 v\,\hat{\mathbf{y}},}$ matching the momentum initially stored in the fields, from part (a). [I thank Michael Ligare for untangling this surprisingly subtle problem. Incidentally, there is also "hidden momentum" in the original configuration. It is not relevant here; it is (relativistic) mechanical momentum (see Example 12.13), and is delivered to the plates as they come together, so it does not affect the overall conservation of momentum.]

Problem 7.59

(a) $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$; $\mathbf{J}$ finite, $\sigma = \infty \Rightarrow \mathbf{E} + (\mathbf{v} \times \mathbf{B}) = 0$. Take the curl: $\nabla \times \mathbf{E} + \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$. But Faraday's law says $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. So $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$. qed

(b) $\nabla \cdot \mathbf{B} = 0 \Rightarrow \oint \mathbf{B} \cdot d\mathbf{a} = 0$ for any closed surface. Apply this at time $(t + dt)$ to the surface consisting of $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{R}$:

$$\int_{\mathcal{S}'} \mathbf{B}(t+dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t+dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t+dt) \cdot d\mathbf{a} = 0$$

(the sign change in the third term comes from switching *outward* $d\mathbf{a}$ to *inward* $d\mathbf{a}$).

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t+dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a} = \int_{\mathcal{S}} \underbrace{[\mathbf{B}(t+dt) - \mathbf{B}(t)]}_{\frac{\partial \mathbf{B}}{\partial t} dt \text{ (for infinitesimal } dt)} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t+dt) \cdot d\mathbf{a}$$

$$d\Phi = \left\{ \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \right\} dt - \int_{\mathcal{R}} \mathbf{B}(t+dt) \cdot [(d\mathbf{l} \times \mathbf{v})\, dt] \quad \text{(Figure 7.13)}.$$

Since the second term is already first order in dt, we can replace $\mathbf{B}(t+dt)$ by $\mathbf{B}(t)$ (the distinction would be second order):

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - dt \oint_{\mathcal{C}} \underbrace{\mathbf{B} \cdot (d\mathbf{l} \times \mathbf{v})}_{(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}} = dt \left\{ \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{a} - \int_{\mathcal{S}} \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{a} \right\}.$$

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right] \cdot d\mathbf{a} = 0. \quad \text{qed}$$

Problem 8.5

(a) $E_x = E_y = 0$, $E_z = -\sigma/\epsilon_0$. Therefore

$$T_{xy} = T_{xz} = T_{yz} = \cdots = 0; \quad T_{xx} = T_{yy} = -\frac{\epsilon_0}{2} E^2 = -\frac{\sigma^2}{2\epsilon_0}; \quad T_{zz} = \epsilon_0 \left(E_z^2 - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} E^2 = \frac{\sigma^2}{2\epsilon_0}.$$

$$\boxed{\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}.}$$

(b) $\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ ($\mathbf{S} = 0$, since $\mathbf{B} = 0$); integrate over the xy plane: $d\mathbf{a} = -dx\, dy\, \hat{\mathbf{z}}$ (negative because *outward* with respect to a surface enclosing the upper plate). Therefore

$$F_z = \int T_{zz}\, da_z = -\frac{\sigma^2}{2\epsilon_0} A, \text{ and the force per unit area is } \mathbf{f} = \frac{\mathbf{F}}{A} = \boxed{-\frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{z}}.}$$

(c) $-T_{zz} = \boxed{\sigma^2/2\epsilon_0}$ is the momentum in the z direction crossing a surface perpendicular to z, per unit area, per unit time (Eq. 8.31).

(d) The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate is

$$\boxed{\mathbf{f} = -\frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{z}}} \quad \text{(same as (b))}.$$

Problem 8.11

(a) From Eq. 5.68 and Prob. 5.36,

$$\begin{cases} r < R: \ \mathbf{E} = 0, \ \mathbf{B} = \dfrac{2}{3}\mu_0 \sigma R \omega \,\hat{\mathbf{z}}, \quad \text{with } \sigma = \dfrac{e}{4\pi R^2}; \\ r > R: \ \mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{e}{r^2}\,\hat{\mathbf{r}}, \ \mathbf{B} = \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \ \text{with } m = \dfrac{4}{3}\pi\sigma\omega R^4. \end{cases}$$

The energy stored in the electric field is (Ex. 2.8):

$$W_E = \frac{1}{8\pi\epsilon_0}\frac{e^2}{R}.$$

The energy density of the internal magnetic field is:

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0 R\omega \frac{e}{4\pi R^2}\right)^2 = \frac{\mu_0 \omega^2 e^2}{72\pi^2 R^2}, \text{ so } W_{B_{in}} = \frac{\mu_0\omega^2 e^2}{72\pi^2 R^2}\frac{4}{3}\pi R^3 = \frac{\mu_0 e^2 \omega^2 R}{54\pi}.$$

The energy density in the external magnetic field is:

$$u_B = \frac{1}{2\mu_0}\frac{\mu_0^2}{16\pi^2}\frac{m^2}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) = \frac{e^2\omega^2 R^4 \mu_0}{18(16\pi^2)}\frac{1}{r^6}\left(3\cos^2\theta + 1\right), \text{ so}$$

$$W_{B_{out}} = \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\int_R^\infty \frac{1}{r^6}r^2\,dr \int_0^\pi \left(3\cos^2\theta + 1\right)\sin\theta\,d\theta \int_0^{2\pi} d\phi = \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\left(\frac{1}{3R^3}\right)(4)(2\pi) = \frac{\mu_0 e^2\omega^2 R}{108\pi}.$$

$$W_B = W_{B_{in}} + W_{B_{out}} = \frac{\mu_0 e^2\omega^2 R}{108\pi}(2+1) = \frac{\mu_0 e^2\omega^2 R}{36\pi}; \ W = W_E + W_B = \boxed{\frac{1}{8\pi\epsilon_0}\frac{e^2}{R} + \frac{\mu_0 e^2 \omega^2 R}{36\pi}.}$$

(b) Same as Prob. 8.8(a), with $Q \to e$ and $m \to \frac{1}{3}e\omega R^2$: $\boxed{\mathbf{L} = \dfrac{\mu_0 e^2 \omega R}{18\pi}\,\hat{\mathbf{z}}.}$

(c) $\dfrac{\mu_0 e^2}{18\pi}\omega R = \dfrac{\hbar}{2} \Rightarrow \omega R = \dfrac{9\pi\hbar}{\mu_0 e^2} = \dfrac{(9)(\pi)(1.05\times 10^{-34})}{(4\pi\times 10^{-7})(1.60\times 10^{-19})^2} = \boxed{9.23\times 10^{10}\,\text{m/s.}}$

$$\frac{1}{8\pi\epsilon_0}\frac{e^2}{R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = mc^2; \ \left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = 1 + \frac{2}{9}\left(\frac{9.23\times 10^{10}}{3\times 10^8}\right)^2 = 2.10\times 10^4;$$

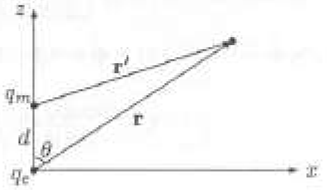
$$R = \frac{(2.01\times 10^4)(1.6\times 10^{-19})^2}{8\pi(8.85\times 10^{-12})(9.11\times 10^{-31})(3\times 10^8)^2} = \boxed{2.95\times 10^{-11}\,\text{m};} \quad \omega = \frac{9.23\times 10^{-10}}{2.95\times 10^{-11}} = \boxed{3.13\times 10^{21}\,\text{rad/s.}}$$

Since ωR, the speed of a point on the equator, is 300 times the speed of light, this "classical" model is clearly unrealistic.

Problem 8.12

$$\mathbf{E} = \frac{q_e}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3};$$

$$\mathbf{B} = \frac{\mu_0 q_m}{4\pi}\frac{\mathbf{r}'}{r'^3} = \frac{\mu_0 q_m}{4\pi}\frac{(\mathbf{r} - d\,\hat{\mathbf{z}})}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}}.$$



Momentum density (Eq. 8.33):

$$\wp = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{(-d)(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\,(r^2+d^2-2rd\cos\theta)^{3/2}}.$$

Angular momentum density (Eq. 8.34):

$$\boldsymbol{\ell} = (\mathbf{r}\times\wp) = -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{\mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\,(r^2+d^2-2rd\cos\theta)^{3/2}}.$$ But $\mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}}) = \mathbf{r}(\mathbf{r}\cdot\hat{\mathbf{z}}) - r^2\,\hat{\mathbf{z}} = r^2\cos\theta\,\hat{\mathbf{r}} - r^2\,\hat{\mathbf{z}}$.

The x and y components will integrate to zero; using $(\hat{\mathbf{r}})_z = \cos\theta$, we have:

$$\begin{aligned}\mathbf{L} &= -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\int\frac{r^2(\cos^2\theta - 1)}{r^3\,(r^2+d^2-2rd\cos\theta)^{3/2}}r^2\sin\theta\,dr\,d\theta\,d\phi. \quad \text{Let } u\equiv\cos\theta:\\ &= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\int_{-1}^{1}\int_0^\infty\frac{r\,(1-u^2)}{(r^2+d^2-2rdu)^{3/2}}\,du\,dr.\end{aligned}$$

Do the r integral first:

$$\int_0^\infty\frac{r\,dr}{(r^2+d^2-2rdu)^{3/2}} = \left.\frac{(ru-d)}{d(1-u^2)\sqrt{r^2+d^2-2rdu}}\right|_0^\infty = \frac{u}{d\,(1-u^2)} + \frac{d}{d\,(1-u^2)\,d} = \frac{u+1}{d\,(1-u^2)} = \frac{1}{d(1-u)}.$$

Then

$$\mathbf{L} = \frac{\mu_0 q_e q_m d}{8\pi}\,\hat{\mathbf{z}}\,\frac{1}{d}\int_{-1}^{1}\frac{(1-u^2)}{(1-u)}\,du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\int_{-1}^{1}(1+u)du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\left(u+\frac{u^2}{2}\right)\Bigg|_{-1}^{1} = \boxed{\frac{\mu_0 q_e q_m}{4\pi}\,\hat{\mathbf{z}}.}$$

Problem 9.7

(a) $F = T\frac{\partial^2 f}{\partial z^2}\Delta z - \gamma\frac{\partial f}{\partial t}\Delta z = \mu\Delta z\frac{\partial^2 f}{\partial t^2}$, or $\boxed{T\frac{\partial^2 f}{\partial z^2} = \mu\frac{\partial^2 f}{\partial t^2} + \gamma\frac{\partial f}{\partial t}.}$

(b) Let $\tilde{f}(z,t) = \tilde{F}(z)e^{-i\omega t}$; then $Te^{-i\omega t}\frac{d^2\tilde{F}}{dz^2} = \mu(-\omega^2)\tilde{F}e^{-i\omega t} + \gamma(-i\omega)\tilde{F}e^{-i\omega t} \Rightarrow$
$T\frac{d^2\tilde{F}}{dz^2} = -\omega(\mu\omega + i\gamma)\tilde{F}$, $\frac{d^2\tilde{F}}{dz^2} = -\tilde{k}^2\tilde{F}$, where $\tilde{k}^2 \equiv \frac{\omega}{T}(\mu\omega + i\gamma)$. Solution : $\tilde{F}(z) = \tilde{A}e^{i\tilde{k}z} + \tilde{B}e^{-i\tilde{k}z}$.

Resolve $\tilde{k}$ into its real and imaginary parts: $\tilde{k} = k + i\kappa \Rightarrow \tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa = \frac{\omega}{T}(\mu\omega + i\gamma)$.
$2k\kappa = \frac{\omega\gamma}{T} \Rightarrow \kappa = \frac{\omega\gamma}{2kT}$; $k^2 - \kappa^2 = k^2 - \left(\frac{\omega\gamma}{2T}\right)^2\frac{1}{k^2} = \frac{\mu\omega^2}{T}$; or $k^4 - k^2(\mu\omega^2/T) - (\omega\gamma/2T)^2 = 0 \Rightarrow$
$k^2 = \frac{1}{2}\left[(\mu\omega^2/T) \pm \sqrt{(\mu\omega^2/T)^2 + 4(\omega\gamma/2T)^2}\right] = \frac{\mu\omega^2}{2T}\left[1 \pm \sqrt{1 + (\gamma/\mu\omega)^2}\right]$. But k is real, so k^2 is positive, so we need the plus sign: $k = \omega\sqrt{\frac{\mu}{2T}}\sqrt{1 + \sqrt{1 + (\gamma/\mu\omega)^2}}$. $\kappa = \frac{\omega\gamma}{2kT} = \frac{\gamma}{\sqrt{2T\mu}}\left[1 + \sqrt{1 + (\gamma/\mu\omega)^2}\right]^{-1/2}$.

Plugging this in, $\tilde{F} = Ae^{i(k+i\kappa)z} + Be^{-i(k+i\kappa)z} = Ae^{-\kappa z}e^{ikz} + Be^{\kappa z}e^{-ikz}$. But the B term gives an exponentially *increasing* function, which we don't want (I assume the waves are propagating in the $+z$ direction), so $B = 0$, and the solution is $\boxed{\tilde{f}(z,t) = \tilde{A}e^{-\kappa z}e^{i(kz-\omega t)}.}$ (The actual displacement of the string is the real part of this, of course.)

(c) The wave is attenuated by the factor $e^{-\kappa z}$, which becomes $1/e$ when
$z = \frac{1}{\kappa} = \boxed{\frac{\sqrt{2T\mu}}{\gamma}\sqrt{1 + \sqrt{1 + (\gamma/\mu\omega)^2}};}$ this is the characteristic penetration depth.

(d) This is the same as before, except that $k_2 \to k + i\kappa$. From Eq. 9.29, $\tilde{A}_R = \left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\tilde{A}_I$;

$$\left(\frac{A_R}{A_I}\right)^2 = \left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\left(\frac{k_1 - k + i\kappa}{k_1 + k - i\kappa}\right) = \frac{(k_1 - k)^2 + \kappa^2}{(k_1 + k)^2 + \kappa^2}. \quad \boxed{A_R = \sqrt{\frac{(k_1 - k)^2 + \kappa^2}{(k_1 + k)^2 + \kappa^2}}A_I}$$

(where $k_1 = \omega/v_1 = \omega\sqrt{\mu_1/T}$, while k and κ are defined in part b). Meanwhile

$$\left(\frac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right) = \frac{(k_1 - k - i\kappa)(k_1 + k + i\kappa)}{(k_1 + k)^2 + \kappa^2} = \frac{(k_1)^2 - k^2 - \kappa^2 - 2i\kappa k_1}{(k_1 + k)^2 + \kappa^2} \Rightarrow \boxed{\delta_R = \tan^{-1}\left(\frac{-2k_1\kappa}{(k_1)^2 - k^2 - \kappa^2}\right).}$$

Problem 9.10

$P = \frac{I}{c} = \frac{1.3 \times 10^3}{3.0 \times 10^8} = \boxed{4.3 \times 10^{-6}\ \text{N/m}^2.}$ For a perfect reflector the pressure is twice as great: $\boxed{8.6 \times 10^{-6}\ \text{N/m}^2.}$ Atmospheric pressure is $1.03 \times 10^5\ \text{N/m}^2$, so the pressure of light on a reflector is $(8.6 \times 10^{-6})/(1.03 \times 10^5) = \boxed{8.3 \times 10^{-11}\ \text{atmospheres.}}$

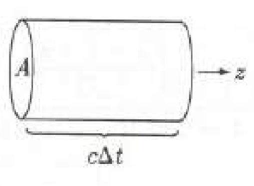
Problem 9.12

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right).$$

With the fields in Eq. 9.48, $\mathbf{E}$ has only an x component, and $\mathbf{B}$ only a y component. So all the "off-diagonal" ($i \neq j$) terms are zero. As for the "diagonal" elements:

$$\begin{aligned}
T_{xx} &= \epsilon_0\left(E_xE_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = \frac{1}{2}\left(\epsilon_0E^2 - \frac{1}{\mu_0}B^2\right) = 0.\\
T_{yy} &= \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_yB_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\right) = 0.\\
T_{zz} &= \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = -u.
\end{aligned}$$

So $\boxed{T_{zz} = -\epsilon_0E_0^2\cos^2(kz - \omega t + \delta)}$ (all other elements zero).



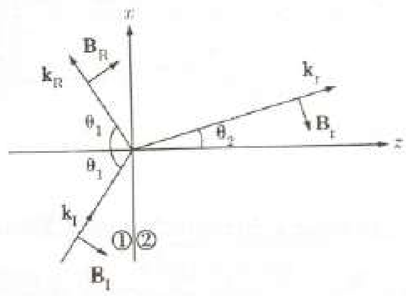
The momentum of these fields is in the z direction, and it is being *transported* in the z direction, so *yes*, it does make sense that T_{zz} should be the only nonzero element in T_{ij}. According to Sect. 8.2.3, $-\overleftrightarrow{T}\cdot d\mathbf{a}$ is the rate at which momentum crosses an area $d\mathbf{a}$. Here we have *no* momentum crossing areas oriented in the x or y direction; the momentum per unit time per unit area flowing across a surface oriented in the z direction is $-T_{zz} = u = \wp c$ (Eq. 9.59), so $\Delta p = \wp cA\Delta t$, and hence $\Delta p/\Delta t = \wp cA =$ momentum per unit time crossing area A. Evidently $\boxed{\text{momentum flux density} = \text{energy density.}}$ ✓

Problem 9.16

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_I &=& \tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_I &=& \dfrac{1}{v_1}\tilde{E}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}(-\cos\theta_1\,\hat{\mathbf{x}}+\sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_R &=& \tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_R &=& \dfrac{1}{v_1}\tilde{E}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}(\cos\theta_1\,\hat{\mathbf{x}}+\sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$

$$\left\{\begin{array}{lcl}\tilde{\mathbf{E}}_T &=& \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}},\\ \tilde{\mathbf{B}}_T &=& \dfrac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_2\,\hat{\mathbf{x}}+\sin\theta_1\,\hat{\mathbf{z}});\end{array}\right\}$$



Boundary conditions: $\left\{\begin{array}{ll}\text{(i) } \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii) } \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel},\\ \text{(ii) } B_1^{\perp} = B_2^{\perp}, & \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}.\end{array}\right.$

Law of refraction: $\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{v_2}{v_1}$. [Note: $\mathbf{k}_I\cdot\mathbf{r}-\omega t = \mathbf{k}_R\cdot\mathbf{r}-\omega t = \mathbf{k}_T\cdot\mathbf{r}-\omega t$, at $z=0$, so we can drop all exponential factors in applying the boundary conditions.]

Boundary condition (i): $0=0$ (trivial). Boundary condition (iii): $\boxed{\tilde{E}_{0_I}+\tilde{E}_{0_R}=\tilde{E}_{0_T}.}$

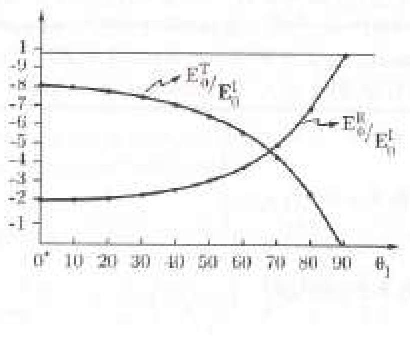
Boundary condition (ii): $\dfrac{1}{v_1}\tilde{E}_{0_I}\sin\theta_1 + \dfrac{1}{v_1}\tilde{E}_{0_R}\sin\theta_1 = \dfrac{1}{v_2}\tilde{E}_{0_T}\sin\theta_2 \Rightarrow \tilde{E}_{0_I}+\tilde{E}_{0_R} = \left(\dfrac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_{0_T}$.
But the term in parentheses is 1, by the law of refraction, so this is the same as (ii).

Boundary condition (iv): $\dfrac{1}{\mu_1}\left[\dfrac{1}{v_1}\tilde{E}_{0_I}(-\cos\theta_1)+\dfrac{1}{v_1}\tilde{E}_{0_R}\cos\theta_1\right] = \dfrac{1}{\mu_2 v_2}\tilde{E}_{0_T}(-\cos\theta_2) \Rightarrow$
$\tilde{E}_{0_I}-\tilde{E}_{0_R} = \left(\dfrac{\mu_1 v_1\cos\theta_2}{\mu_2 v_2\cos\theta_1}\right)\tilde{E}_{0_T}$. Let $\boxed{\alpha\equiv\dfrac{\cos\theta_2}{\cos\theta_1};\ \beta\equiv\dfrac{\mu_1 v_1}{\mu_2 v_2}.}$ Then $\boxed{\tilde{E}_{0_I}-\tilde{E}_{0_R}=\alpha\beta\tilde{E}_{0_T}.}$

Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$: $2\tilde{E}_{0_I} = (1+\alpha\beta)\tilde{E}_{0_T} \Rightarrow \tilde{E}_{0_T} = \left(\dfrac{2}{1+\alpha\beta}\right)\tilde{E}_{0_I}$;
$\tilde{E}_{0_R} = \tilde{E}_{0_T}-\tilde{E}_{0_I} = \left(\dfrac{2}{1+\alpha\beta}-\dfrac{1+\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I} \Rightarrow \tilde{E}_{0_R} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0_I}$.

Since α and β are positive, it follows that $2/(1+\alpha\beta)$ is positive, and hence the *transmitted* wave is *in phase* with the incident wave, and the (real) amplitudes are related by $\boxed{E_{0_T} = \left(\dfrac{2}{1+\alpha\beta}\right)E_{0_I}.}$ The *reflected* wave is in phase if $\alpha\beta<1$ and 180° out of phase if $\alpha\beta<1$; the (real) amplitudes are related by $\boxed{E_{0_R} = \left|\dfrac{1-\alpha\beta}{1+\alpha\beta}\right|E_{0_I}.}$
These are the **Fresnel equations** for polarization perpendicular to the plane of incidence.

To construct the graphs, note that $\alpha\beta = \beta\dfrac{\sqrt{1-\sin^2\theta/\beta^2}}{\cos\theta} = \dfrac{\sqrt{\beta^2-\sin^2\theta}}{\cos\theta}$, where θ is the angle of incidence, so, for $\beta=1.5$, $\alpha\beta = \dfrac{\sqrt{2.25-\sin^2\theta}}{\cos\theta}$.



Is there a Brewster's angle? Well, $E_{0_R} = 0$ would mean that $\alpha\beta = 1$, and hence that

$\alpha = \dfrac{\sqrt{1-(v_2/v_1)^2\sin^2\theta}}{\cos\theta} = \dfrac{1}{\beta} = \dfrac{\mu_2 v_2}{\mu_1 v_1}$, or $1 - \left(\dfrac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\dfrac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta$, so $1 = \left(\dfrac{v_2}{v_1}\right)^2 \left[\sin^2\theta + (\mu_2/\mu_1)^2\cos^2\theta\right]$. Since $\mu_1 \approx \mu_2$, this means $1 \approx (v_2/v_1)^2$, which is only true for optically indistinguishable media, in which case there is of *course* no reflection—but that would be true at *any* angle, not just at a special "Brewster's angle". [If μ_2 were substantially different from μ_1, and the relative velocities were just right, it *would* be possible to get a Brewster's angle for this case, at

$$\left(\frac{v_1}{v_2}\right)^2 = 1 - \cos^2\theta + \left(\frac{\mu_2}{\mu_1}\right)^2 \cos^2\theta \Rightarrow \cos^2\theta = \frac{(v_1/v_2)^2 - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\mu_2\epsilon_2/\mu_1\epsilon_1) - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\epsilon_2/\epsilon_1) - (\mu_1/\mu_2)}{(\mu_2/\mu_1) - (\mu_1/\mu_2)}.$$

But the media would be very peculiar.]

By the same token, δ_R is either always 0, or always π, for a given interface—it does not switch over as you change θ, the way it does for polarization in the plane of incidence. In particular, if $\beta = 3/2$, then $\alpha\beta > 1$, for

$$\alpha\beta = \frac{\sqrt{2.25 - \sin^2\theta}}{\cos\theta} > 1 \text{ if } 2.25 - \sin^2\theta > \cos^2\theta, \text{ or } 2.25 > \sin^2\theta + \cos^2\theta = 1. \checkmark$$

In general, for $\beta > 1$, $\alpha\beta > 1$, and hence $\delta_R = \pi$. For $\beta < 1$, $\alpha\beta < 1$, and $\delta_R = 0$.

At normal incidence, $\alpha = 1$, so Fresnel's equations reduce to $E_{0_T} = \left(\dfrac{2}{1+\beta}\right) E_{0_I}$; $E_{0_R} = \left|\dfrac{1-\beta}{1+\beta}\right| E_{0_I}$, consistent with Eq. 9.82.

Reflection and Transmission coefficients: $\boxed{R = \left(\dfrac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2.}$ Referring to Eq. 9.116,

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \boxed{\alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2.}$$

$$R + T = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1. \checkmark$$

Problem 9.20

(a) $u = \dfrac{1}{2}\left(\epsilon E^2 + \dfrac{1}{\mu}B^2\right) = \dfrac{1}{2}e^{-2\kappa z}\left[\epsilon E_0^2\cos^2(kz-\omega t+\delta_E) + \dfrac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\phi)\right]$. Averaging over a full cycle, using $\langle\cos^2\rangle = \frac{1}{2}$ and Eq. 9.137:

$$\langle u\rangle = \frac{1}{2}e^{-2\kappa z}\left[\frac{\epsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right] = \frac{1}{4}e^{-2\kappa z}\left[\epsilon E_0^2 + \frac{1}{\mu}E_0^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] = \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left[1 + \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right].$$

But Eq. 9.126 $\Rightarrow 1 + \sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2} = \dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2}$, so $\langle u\rangle = \dfrac{1}{4}e^{-2\kappa z}\epsilon E_0^2\dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2} = \boxed{\dfrac{k^2}{2\mu\omega^2}E_0^2 e^{-2\kappa z}.}$ So the ratio of the magnetic contribution to the electric contribution is

$$\frac{\langle u_{\text{mag}}\rangle}{\langle u_{\text{elec}}\rangle} = \frac{B_0^2/\mu}{E_0^2\epsilon} = \frac{1}{\mu\epsilon}\mu\epsilon\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} = \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} > 1. \quad \text{qed}$$

(b) $\mathbf{S} = \dfrac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = \dfrac{1}{\mu}E_0B_0e^{-2\kappa z}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{z}}$; $\langle\mathbf{S}\rangle = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi\,\hat{\mathbf{z}}$. [The average of the product of the cosines is $(1/2\pi)\int_0^{2\pi}\cos\theta\cos(\theta+\phi)\,d\theta = (1/2)\cos\phi$.] So $I = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi = \dfrac{1}{2\mu}E_0^2e^{-2\kappa z}\left(\dfrac{K}{\omega}\cos\phi\right)$, while, from Eqs. 9.133 and 9.134, $K\cos\phi = k$, so $\boxed{I = \dfrac{k}{2\mu\omega}E_0^2e^{-2\kappa z}.}$ qed

Problem 9.30

Following Sect. 9.5.2, the problem is to solve Eq. 9.181 with $E_z \neq 0, B_z = 0$, subject to the boundary conditions 9.175. Let $E_z(x,y) = X(x)Y(y)$; as before, we obtain $X(x) = A\sin(k_x x) + B\cos(k_x x)$. But the boundary condition requires $E_z = 0$ (and hence $X = 0$) when $x = 0$ and $x = a$, so $B = 0$ and $k_x = m\pi/a$. But this time $m = 1, 2, 3, \ldots$, but *not* zero, since $m = 0$ would kill X entirely. The same goes for $Y(y)$. Thus $\boxed{E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$ with $n, m = 1, 2, 3, \ldots$.

The rest is the same as for TE waves: $\boxed{\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}}$ is the cutoff frequency, the wave velocity is $v = c/\sqrt{1 - (\omega_{mn}/\omega)^2}$, and the group velocity is $v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$. The lowest TM mode is 11, with cutoff frequency $\omega_{11} = c\pi\sqrt{(1/a)^2 + (1/b)^2}$. So the ratio of the lowest TM frequency to the lowest TE frequency is $\dfrac{c\pi\sqrt{(1/a)^2 + (1/b)^2}}{(c\pi/a)} = \boxed{\sqrt{1 + (a/b)^2}.}$

Problem 9.31

(a) $\nabla\cdot\mathbf{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) = 0 \checkmark$; $\nabla\cdot\mathbf{B} = \frac{1}{s}\frac{\partial}{\partial\phi}(B_\phi) = 0 \checkmark$; $\nabla\times\mathbf{E} = \frac{\partial E_s}{\partial z}\,\hat{\boldsymbol{\phi}} - \frac{1}{s}\frac{\partial E_s}{\partial\phi}\,\hat{\mathbf{z}} = -\frac{E_0 k\sin(kz-\omega t)}{s}\,\hat{\boldsymbol{\phi}} \stackrel{?}{=} -\frac{\partial\mathbf{B}}{\partial t} = -\frac{E_0\omega}{c}\frac{\sin(kz-\omega t)}{s}\,\hat{\boldsymbol{\phi}} \checkmark$ (since $k = \omega/c$); $\nabla\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\,\hat{\mathbf{z}} = \frac{E_0 k}{c}\frac{\sin(kz-\omega t)}{s}\,\hat{\mathbf{s}} \stackrel{?}{=} \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \frac{E_0\omega}{c^2}\frac{\sin(kz-\omega t)}{s}\,\hat{\mathbf{s}} \checkmark$. Boundary conditions: $E^{\parallel} = E_z = 0 \checkmark$; $B^{\perp} = B_s = 0 \checkmark$.

(b) To determine λ, use Gauss's law for a cylinder of radius s and length dz:
$\oint \mathbf{E}\cdot d\mathbf{a} = E_0\frac{\cos(kz-\omega t)}{s}(2\pi s)\,dz = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\lambda\,dz \Rightarrow \boxed{\lambda = 2\pi\epsilon_0 E_0\cos(kz-\omega t).}$

To determine I, use Ampére's law for a circle of radius s (note that the displacement current through this loop is zero, since $\mathbf{E}$ is in the $\hat{\mathbf{s}}$ direction): $\oint \mathbf{B}\cdot d\mathbf{l} = \frac{E_0}{c}\frac{\cos(kz-\omega t)}{s}(2\pi s) = \mu_0 I_{\text{enc}} \Rightarrow \boxed{I = \frac{2\pi E_0}{\mu_0 c}\cos(kz-\omega t).}$

The charge and current on the outer conductor are precisely the $\boxed{\text{opposite}}$ of these, since $\mathbf{E} = \mathbf{B} = 0$ *inside* the metal, and hence the *total* enclosed charge and current must be zero.

Problem 9.33

(a) (i) *Gauss's law:* $\nabla\cdot\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0.\ \checkmark$

(ii) *Faraday's law:*

$$\begin{aligned}
-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\,\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[E_0\frac{\sin^2\theta}{r}\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left[E_0\sin\theta\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{\boldsymbol{\theta}}.
\end{aligned}$$

But $\frac{\partial}{\partial r}\cos u = -k\sin u$; $\frac{\partial}{\partial r}\sin u = k\cos u$.

$$= \frac{1}{r\sin\theta}\frac{E_0}{r}2\sin\theta\cos\theta\left(\cos u - \frac{1}{kr}\sin u\right)\hat{\mathbf{r}} - \frac{1}{r}E_0\sin\theta\left(-k\sin u + \frac{1}{kr^2}\sin u - \frac{1}{r}\cos u\right)\hat{\boldsymbol{\theta}}.$$

Integrating with respect to t, and noting that $\int \cos u\,dt = -\frac{1}{\omega}\sin u$ and $\int \sin u\,dt = \frac{1}{\omega}\cos u$, we obtain

$$\boxed{\mathbf{B} = \frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\mathbf{r}} + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\hat{\boldsymbol{\theta}}.}$$

(iii) *Divergence of* $\mathbf{B}$:

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2E_0\cos\theta}{\omega}\left(\sin u + \frac{1}{kr}\cos u\right)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{E_0\sin^2\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] \\
&= \frac{1}{r^2}\frac{2E_0\cos\theta}{\omega}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{2E_0\sin\theta\cos\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)
\end{aligned}$$

$$= \frac{2E_0\cos\theta}{\omega r^2}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right) = 0. \checkmark$$

(iv) *Ampére/Maxwell:*

$$\begin{aligned}
\nabla\times\mathbf{B} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi} \\
&= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{E_0\sin\theta}{\omega}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\right]\right\}\hat{\phi} \\
&= \frac{E_0\sin\theta}{\omega r}\left(k^2\sin u - \frac{2}{kr^3}\cos u - \frac{1}{r^2}\sin u - \frac{1}{r^2}\sin u + \frac{k}{r}\cos u + \frac{2}{r^2}\sin u + \frac{2}{kr^3}\cos u\right)\hat{\phi} \\
&= \frac{k}{\omega}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi}. \\
\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{1}{c^2}\frac{E_0\sin\theta}{r}\left(\omega\sin u + \frac{\omega}{kr}\cos u\right)\hat{\phi} = \frac{1}{c^2}\frac{\omega}{k}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} \\
&= \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \nabla\times\mathbf{B}. \checkmark
\end{aligned}$$

(b) *Poynting Vector:*

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{E_0\sin\theta}{\mu_0 r}\left(\cos u - \frac{1}{kr}\sin u\right)\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\theta}\right. \\
&\quad \left. + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)(-\hat{\mathbf{r}})\right] \\
&= \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u) - \frac{1}{k^2r^2}\sin u\cos u\right]\hat{\theta}\right. \\
&\quad \left. - \sin\theta\left(-k\cos^2 u + \frac{1}{kr^2}\cos^2 u + \frac{1}{r}\sin u\cos u + \frac{1}{r}\sin u\cos u - \frac{1}{k^2r^3}\sin u\cos u - \frac{1}{kr^2}\sin^2 u\right)\hat{\mathbf{r}}\right\} \\
&= \boxed{\begin{aligned}&\frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u)\right]\hat{\theta}\right. \\
&\quad \left. + \sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}(\sin^2 u - \cos^2 u)\right]\hat{\mathbf{r}}\right\}.\end{aligned}}
\end{aligned}$$

Averaging over a full cycle, using $\langle\sin u\cos u\rangle = 0$, $\langle\sin^2 u\rangle = \langle\cos^2 u\rangle = \frac{1}{2}$, we get the intensity:

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left(\frac{k}{2}\sin\theta\right)\hat{\mathbf{r}} = \boxed{\frac{E_0^2\sin^2\theta}{2\mu_0 c r^2}\hat{\mathbf{r}}.}$$

It points in the $\hat{\mathbf{r}}$ direction, and falls off as $1/r^2$, as we would expect for a spherical wave.

(c) $P = \int \mathbf{I}\cdot d\mathbf{a} = \dfrac{E_0^2}{2\mu_0 c}\displaystyle\int \frac{\sin^2\theta}{r^2}r^2\sin\theta\, d\theta\, d\phi = \frac{E_0^2}{2\mu_0 c}2\pi\int_0^\pi \sin^3\theta\, d\theta = \boxed{\frac{4\pi}{3}\frac{E_0^2}{\mu_0 c}.}$